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New Aspects in Bifurcation with Symmetry

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1. Introduction

It comes as no surprise that the literature devoted to bifurcation problems involving symmetry draws from both analysis and group theory. However, it is more accurate to say that it draws from both analysis and group representation theory, the two being related through the notion of isotropy subgroup. Isotropy subgroups have been crucial to every work having some connection with bifurcation and symmetry. To give only one example, they are a central theme in the recent volume by Golubitsky, Stewart and Schaeffer [7].

Given a group Γ and a representation R of Γ in $GL(R^n)$, recall that the isotropy subgroup of $x \in \mathbf{R}^n$ relative to R is the subgroup of $\Gamma:\{\gamma \in \Gamma; R_\gamma x = x\}$ (here and everywhere else in this paper, R_γ stands for $R(\gamma)$). This definition makes it clear that isotropy subgroups are strongly representation dependent.

It is the aim of this paper to show that there are connections between bifurcation problems and group theory that do not refer to isotropy subgroups and are virtually representation-independent. In particular, it will be shown that Lie groups, including finite ones, possess subgroups of a special type, here called intrinsic isotropy subgroups, characterized by a representation-independent property, which play a role in bifurcation problems. To be more precise, consider a bifurcation equation $g(\mu,x) = 0$ with g acting from $\mathbf{R}\times\mathbf{R}^n$ to $\mathbf{R}^n$ with $g(\mu,0) \equiv 0$ and suppose that det $D_x g(\mu,0)$ changes sign as μ crosses 0. Suppose also that $g(\mu,\cdot)$ is covariant under some orthogonal representation R of a Lie group Γ. Orthogonality of R is never a restrictive assumption when Γ is compact. Then, Theorem 2.1 of the next section ascertains that whenever $\Gamma' \subset \Gamma$ is an intrinsic isotropy subgroup of Γ, a continuum of Γ'-symmetric nontrivial solutions to $g = 0$ bifurcates from (0,0).

While the proof of Theorem 2.1 is straightforward from the very definition of intrinsic isotropy subgroups and a topological degree argument quite standard in these matters, its value is due to a much less evident result of existence of nontrivial intrinsic isotropy subgroups, the trivial case being that of $\Gamma' = \{1\}$. Although some Lie groups such as **R** or **Z** do not possess any nontrivial intrinsic isotropy subgroup, it is rather striking that all compact Lie groups do, with the only exception of groups isomorphic to a direct product of copies of Z_2, hence finite. This is proved in Section 3 as a corollary to a more general theorem (Theorem 3.1). Along with other results establishing general properties, Theorem 3.1 also allows us to give specific examples presented in Section 4.

For instance, we show that every finite subgroup with odd order is an intrinsic isotropy subgroup. Another especially interesting result is that maximal tori of compact Lie groups *are intrinsic* isotropy subgroups, but not maximal ones in general (an exception being SO(2) in SO(3)). It does not seem that the intrinsic isotropy subgroups of this paper have been studied by group theory specialists.

In Section 5, we have briefly discussed the consequences of Theorem 2.1 in the perspective of the examples of Section 4. Indeed, these consequences call for a comparison with other theorems in the literature, obtained under hypotheses involving the standard representation - dependent isotropy subgroups.

As mentioned earlier, this paper intends to discuss some relationships between bifurcation problems and group theory making no reference to such or such specific property of the representation. Intrinsic isotropy subgroups are not the only way to achieve this goal: in Section 6, we prove a theorem

complementing Theorem 2.1 when the group Γ of interest is $Z_2 \times Z_2$ (or any group containing $Z_2 \times Z_2$). As $Z_2 \times Z_2$ has no nontrivial intrinsic isotropy subgroup, Theorem 2.1 is meaningless in this case. Another interesting aspect of the theorem of Section 6 is that, unlike Theorem 2.1, it can be extended to variational problems if the condition about change of sign of the determinant is replaced by a more general one where change of Morse index only is required. This is shown in Section 7, and leads to the problem of finding a general framework, if any, for the validity of such theorems.

Finally, in Section 8, we make a few comments regarding the availability of the results obtained earlier for bifurcation problems in Banach spaces.

Throughout this paper, multiplication is understood to be the group operation and 1 denotes the identity element of all abstract groups. All Lie groups are supposed to be real. As usual, when Γ is a Lie group and R is a representation of Γ in $GL(\mathbf{R}^n)$, it is implicit that $\gamma \in \Gamma \to R_\gamma \in GL(\mathbf{R}^n)$ is continuous. Given $\gamma \in \Gamma$, the subspace $X_\gamma = \{x \in \mathbf{R}^n;\ R_\gamma x = x\}$ will be called the fixed point space of γ relative to R. Similarly, if $H \subset \Gamma$ is any subset, the subspace $X_H = \bigcap_{\gamma \in H} X_\gamma = \{x \in \mathbf{R}^n;\ R_\gamma x = x,\ \forall \gamma \in H\}$, will be called the fixed point space of H relative to R. Mappings (linear or not) commuting with R are said to be R-covariant, or covariant under R. More generally, if two representations R and S of Γ are given in $GL(\mathbf{R}^n)$, a mapping $f : \mathbf{R}^n \to \mathbf{R}^n$ such that $f(R_\gamma x) = S_\gamma f(x)$, $\forall x \in \mathbf{R}^n$, $\forall \gamma \in \Gamma$, will be called (R,S)-covariant. We shall be concerned only with the case when R and S are equivalent, that is when a (R,S)-covariant linear isomorphism $A \in GL(\mathbf{R}^n)$ exists.

2. Intrinsic isotropy subgroups and bifurcation.

Let Γ be any Lie group and let R denote a representation of Γ in $GL(R^n)$. Given a subgroup $\Gamma' \subset \Gamma$, recall that $X_{\Gamma'}$ is the fixed point space of Γ' relative to R. If now $L \in \mathcal{L}(R^n)$ is R-covariant, it is trivial that $L(X_{\Gamma'}) \subset X_{\Gamma'}$. Moreover, if $L \in GL(R^n)$, then it is equally obvious that $L_{\Gamma'} \equiv L_{|X_{\Gamma'}} \in GL(X_{\Gamma'})$. With these preliminary remarks, we can now make the following definition:

Definition 2.1: *We shall say that the subgroup $\Gamma' \subset \Gamma$ is an intrinsic isotropy subgroup (i.i.s. for short) if, given any integer $n \geq 1$, any representation R of Γ in $GL(R^n)$ and any R-covariant linear isomorphism $L \in GL(R^n)$, one has*

$$\operatorname{sgn} \det L_{\Gamma'} = \operatorname{sgn} \det L \;.\square$$

With the convention that the sign of the determinant of the unique isomorphism of the trivial space $\{0\}$ is positive, the above definition contains the implicit condition that $X_{\Gamma'} \neq \{0\}$ whenever R is a representation of Γ such that a R-covariant linear isomorphism $L \in GL(R^n)$ exists with $\det L < 0$. Such an L, however, does not always exist (example: $\Gamma = SO(2)$, $n = 2$ and R is the standard representation of $SO(2)$ in $GL(R^2)$). Obviously, $\Gamma' = \{1\}$ is an intrinsic isotropy subgroup since $X_{\{1\}} = R^n$.

Intrinsic isotropy subgroups have a lot in common with isotropy subgroups relative to a given representation R. In particular, any conjugate of an i.i.s. is an i.i.s. More generally, the image of an i.i.s. under any automorphism is an i.i.s. This follows easily from the remark that for every automorphism a of Γ and every representation R of Γ, the mapping R^a defined by

$R^a_\gamma = R_{a(\gamma)}$ is a representation of Γ admitting the same covariant mappings as R. Also, one may define <u>maximal intrinsic isotropy subgroups</u> (m.i.i.s. for short) in the obvious way, thus saying that $\Gamma' \subset \Gamma$ is a m.i.i.s. of Γ if it is an i.i.s. which is contained in no other i.i.s. A straightforward application of Zorn's lemma via Lemma 3.2 of the next section will show that every i.i.s. is contained in at least one m.i.i.s.

Now, suppose that the Lie group Γ is given along with one of its representations R in $GL(R^n)$. We shall consider a $\mathcal{C}^1$ mapping $g(=g(\mu,x))$: $R\times R^n \to R^n$ such that $g(\mu,0) = 0$ and assume that, for every $\mu \in R$, $g(\mu,\cdot)$is R-covariant. We shall also assume that $\det D_x g(\mu,0)$ <u>changes sign</u> as μ crosses 0. Implicit here is the fact that $D_x g(\mu,0) \in GL(R^n)$ for $|\mu| > 0$ small enough. If $\Gamma' \subset \Gamma$ is a subgroup, we shall say that the point $(\mu,x) \in R\times R^n$ has Γ'-symmetry if $x \in X_{\Gamma'}$, the fixed point space of Γ' relative to R. With the above hypotheses, one has

<u>Theorem 2.1</u>: <u>Let $\Gamma' \subset \Gamma$ be a (maximal) intrinsic isotropy subgroup. Then, there is a continuum of nontrivial solutions to $g = 0$ with Γ'-symmetry that bifurcates from $(0,0)$.</u>

<u>Proof</u>: Since $\det D_x g(\mu,0)$ changes sign as μ crosses 0, one can find μ_0 such that $\det D_x g(\mu_0,0) < 0$. As $D_x g(\mu,0)$ is R-covariant and Γ' is an i.i.s., this yields that $X_{\Gamma'} \neq \{0\}$ and, further, that $\det D_x g(\mu,0)|_{X_{\Gamma'}}$ changes sign as μ crosses 0. On the other hand, $D_x g(\mu,0)|_{X_{\Gamma'}}$ is the x-derivative at $(\mu,0)$ of the restriction of $g(\mu,\cdot)$ to the space $X_{\Gamma'}$ (that $g(\mu,\cdot)$ maps the space $X_{\Gamma'}$ into itself is due to R-covariance). The conclusion follows from the usual topological degree argument applied with $g : R\times X_{\Gamma'} \to X_{\Gamma'}$ (see e.g. Chow and Hale [3] for the case when g is $\mathcal{C}^2$; that the result remains true when g is $\mathcal{C}^1$

is well known). □

Remark 2.1: Of course, Theorem 2.1 is valid when g is only defined in the vicinity of $(0,0)$. It is also valid when the action of Γ is different in the source and target spaces R^n, and hence accounted for by two equivalent representations R and S (equivalence is necessary for consistency with the hypothesis that $D_x g(\mu,0) \in GL(R^n)$ for $|\mu| > 0$ small enough). Indeed, with $A \in GL(R^n)$ being such that $AR_\gamma = S_\gamma A$, $\forall \gamma \in \Gamma$, the mapping $A^{-1}g : R \times R^n \to R^n$ fulfills all the conditions required of g in Theorem 2.1, R-covariance in particular. □

The value of Theorem 2.1 relies only in the fact that there are Lie groups having nontrivial intrinsic isotropy subgroups. Some, however, do not. Take for instance $\Gamma = R_+$ with representation in $GL(R^n)$ through scalar multiplication: $R_\gamma x = \gamma x$, $\forall \gamma \in R_+$, $\forall x \in R^n$. Every $L \in GL(R^n)$ is R-covariant. In particular, there are covariant isomorphisms with negative determinant. But given $\gamma \in R_+$, $\gamma \neq 1$, it is obvious that $X_\gamma = \{x \in R^n;\ \gamma x = x\} = \{0\}$, so that $X_{\Gamma'} = \{0\}$ for every subgroup $\Gamma' \subset R_+$ with $\Gamma' \neq \{1\}$. Thus, no nontrivial subgroup of R_+ is an i.i.s. Also, $\Gamma = Z_2$ has no nontrivial i.i.s. (let $Z_2 = \{-1,1\}$ and consider the representation $R_1 = I$, $R_{-1} = -I$ in $GL(R^n)$) and neither does $\Gamma = Z_2 \times \cdots \times Z_2$ (see Section 3).

3. Existence and general properties of intrinsic isotropy subgroups.

Our first task will be to prove that, in any Lie group Γ, a given i.i.s. Γ' is contained in at least one m.i.i.s. This will follow from Zorn's lemma through the elementary

Lemma 3.1: *let A be any nonempty set and X_α, $\alpha \in A$, a family of subspaces of*

R^n. Set $X = \bigcap_{\alpha \in A} X_\alpha$. Then, there is an integer $k \geq 1$ and elements α_j, $1 \leq j \leq k$, such that $X = \bigcap_{j=1}^{k} X_{\alpha_j}$.

Proof: We assume that A is infinite, for the result is trivial otherwise. Let $\alpha_1 \in A$ be arbitrary. Since $X \subset X_{\alpha_1}$, the proof is complete if equality holds. If not, let $x_1 \in X_{\alpha_1}$, $x_1 \notin X$, so that there is $\alpha_2 \in A$ such that $x_1 \notin X_{\alpha_2}$. Clearly, $X \subset X_{\alpha_1} \cap X_{\alpha_2}$ and $\dim(X_{\alpha_1} \cap X_{\alpha_2}) < \dim X_{\alpha_1}$. If $X = X_{\alpha_1} \cap X_{\alpha_2}$, the proof is complete. Proceeding inductively as above, it is plain that if the lemma is not true one may find an infinite sequence $\alpha_j \in A$ such that $\dim(X_{\alpha_1} \cap \cdots \cap X_{\alpha_j})$ is strictly decreasing, which is absurd. □

Proposition 3.1: let Γ be any Lie group and $\Gamma' \subset \Gamma$ an i.i.s. Then, there is a m.i.i.s. of Γ containing Γ'. In particular, every Lie group contains a m.i.i.s.

Proof: One must show that the set of all intrinsic isotropy subgroups containing Γ' possesses a maximal element relative to $\subset$. This set is not empty (it contains Γ') and, by Zorn's lemma, it suffices to show that every subset $(\Gamma'_\alpha)_{\alpha \in A}$ where A is an index set such that $\Gamma'_\alpha \subset \Gamma'_\beta$ or $\Gamma'_\beta \subset \Gamma'_\alpha$ for every pair $\alpha, \beta \in A$, has an upper bound. In fact, $\tilde{\Gamma}' = \bigcup_{\alpha \in A} \Gamma'_\alpha$ can be taken as an upper bound. Indeed, $\tilde{\Gamma}'$ is obviously a subgroup of Γ and, given a representation R of Γ in $GL(R^n)$, one has

$$X_{\tilde{\Gamma}'} = \bigcap_{\alpha \in A} X_{\Gamma'_\alpha} ,$$

where $X_{\tilde{\Gamma}'}$ (resp. $X_{\Gamma'_\alpha}$) denotes the fixed point space of $\tilde{\Gamma}'$(resp. Γ'_α) relative to R. Using Lemma 3.1, one finds $\alpha_1, \ldots, \alpha_k \in A$ such that, setting $X_j = X_{\Gamma'_{\alpha_j}}$ for simplicity of notation,

$$X_{\tilde{\Gamma}'} = \bigcap_{j=1}^{k} X_j \ .$$

Now, for $1 \leq j, \ell \leq k$, one has $\Gamma'_{\alpha_j} \subset \Gamma'_{\alpha_\ell}$ or $\Gamma'_{\alpha_\ell} \subset \Gamma'_{\alpha_j}$ by hypothesis, and hence $X_\ell \subset X_j$ or $X_j \subset X_\ell$, respectively. This yields that $\bigcap_{j=1}^{k} X_j = X_{j_0}$ for some index $1 \leq j_0 \leq k$, namely $X_{\tilde{\Gamma}'} = X_{j_0}$. Let then $L \in GL(R^n)$ be a R-covariant linear isomorphism and set $L_{\tilde{\Gamma}'} = L_{|X_{\tilde{\Gamma}'}}$. That sgn $\det L_{\tilde{\Gamma}'} =$ sgn $\det L$ follows from $X_{\tilde{\Gamma}'} = X_{j_0}$ and the fact that $\Gamma'_{\alpha_{j_0}}$ is an i.i.s. of Γ by hypothesis, and this shows that $\tilde{\Gamma}'$ is an i.i.s.

The second assertion of the lemma follows from the first one with $\Gamma' = \{1\}$, and the proof is complete. □

We shall prove existence of nontrivial intrinsic isotropy subgroups of compact Lie groups by proving existence of nontrivial maximal ones. This, of course, does not follow from Proposition 3.1. Before, we need a lemma in linear algebra.

<u>Lemma 3.2</u>: <u>let</u> $m \geq 2$ <u>be an integer and</u> $Q \in \mathcal{L}(R^m)$ <u>an orthogonal operator such that</u> $\pm 1 \notin$ Sp Q (<u>so that</u> m <u>is even</u>). <u>Then, if</u> $L \in \mathcal{L}(R^m)$ <u>commutes with</u> Q, <u>one has</u>

$$\det L \geq 0 \ .$$

<u>Proof</u>: For convenience, we identify Q and L with their matrices in the canonical basis of R^m.

Since $\pm 1 \notin$ Sp Q, the eigenvalues of Q occur in conjugate complex pairs with norm 1. In particular, Q is a rotation ($\det Q = 1$) and hence has the

block diagonal real Jordan form (see e.g. [13])

$$J = \operatorname{diag}\,(D_1,\ldots,D_k) \tag{3.1}$$

where $k = m/2$ and D_k is a 2×2 matrix given by (plane rotation)

$$D_k = \begin{pmatrix} \cos\theta_k & -\sin\theta_k \\ \sin\theta_k & \cos\theta_k \end{pmatrix},\ \theta_k \neq 0 \bmod \pi\ . \tag{3.2}$$

As Q is similar to J through a real transformation, say $Q = U^{-1}JU$, it suffices to change L into ULU^{-1} to reduce the problem to the case when $Q = J$. Now, we proceed by induction on k. If $k = 1$, i.e. $m=2$, then $J = D_1$ and a 2×2 matrix L commuting with D_1 has the form (using $\theta_1 \neq 0 \bmod \pi$; see (3.2))

$$L = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix},$$

whose determinant $\alpha^2 + \beta^2$ is nonnegative.

In the general case, we use the block decomposition

$$L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix},\ J = \begin{pmatrix} D_1 & 0 \\ 0 & J_2 \end{pmatrix},$$

where $J_2 = \operatorname{diag}(D_2,\ldots,D_k)$ is $(2k-2)\times(2k-2)$. Naturally, the block decomposition of L is chosen consistently. Writing $LQ = QL$, one finds the equivalent set of relations

$$L_{11}D_1 = D_1L_{11}\ , \tag{3.3}$$

$$L_{12}J_2 = D_1L_{12}\ , \tag{3.4}$$

$$L_{21}D_1 = J_2L_{21}\ , \tag{3.5}$$

$$L_{22}J_2 = J_2L_{22}\ . \tag{3.6}$$

Suppose now that L_{11} is invertible. If so, it is a standard result that

$$\det L = (\det L_{11}) \det(L_{22} - L_{21}L_{11}^{-1}L_{12}) .$$

That $\det L_{11} \geq 0$ follows from (3.3) and the first part of the proof. Similarly, the hypothesis of induction yields that $\det(L_{22} - L_{21}L_{11}^{-1}L_{12}) \geq 0$ if one can show that $L_{22} - L_{21}L_{11}^{-1}L_{12}$ and J_2 commute (obviously, J_2 is orthogonal and $\pm 1 \notin \mathrm{Sp}\, J_2$). But this follows readily from relations (3.3) to (3.6).

If L_{11} is not invertible, then $L_{11} + \epsilon I_1 (I_1 =$ identity of $R^2)$ is invertible for $\epsilon > 0$ small enough. Thus, the matrix $L_\epsilon = L + \epsilon I$ commutes with J, and $\det L_\epsilon \geq 0$ from the above. Finally, $\det L \geq 0$ by continuity of $\det L_\epsilon$ w.r.t. ϵ.
□

The next and final lemma contains most of the substance of our proof of existence of nontrivial intrinsic isotropy subgroups in Corollary 3.1 after Theorem 3.1 below.

<u>Lemma 3.3</u>: <u>Let Γ be a compact Lie group. Then, given any i.i.s. Γ' of Γ and any $\gamma \in N_\Gamma(\Gamma')$ - the normalizer of Γ' in Γ - the subgroup $\langle\gamma^2,\Gamma'\rangle \subset \Gamma$ generated by γ^2 and Γ' is also an i.i.s. of Γ.</u>

<u>Proof</u>: Because every representation of Γ in $GL(R^n)$ is equivalent to an orthogonal representation, it is easily checked that one may confine attention to those representations that are orthogonal in the definition of intrinsic isotropy subgroups (Definition 2.1). Accordingly, we henceforth assume that R is an orthogonal representation of Γ in $GL(R^n)$.

Let $\tilde{\Gamma}' = \langle\gamma^2,\Gamma'\rangle$. Given a R-covariant linear isomorphism $L \in GL(R^n)$, we must show that $\mathrm{sgn}\, \det L_{\tilde{\Gamma}'} = \mathrm{sgn}\, \det L$, where $L_{\tilde{\Gamma}'} \equiv L_{|X_{\tilde{\Gamma}'}}$ and $X_{\tilde{\Gamma}'}$ is the fixed

point space of $\tilde{\Gamma}'$ relative to R(in particular, it must be shown that $X_{\tilde{\Gamma}'} \neq \{0\}$ if such an L exists with $\det L < 0$). Because Γ' is an i.i.s., this property is true with Γ' replacing $\tilde{\Gamma}'$ above.

In a preliminary step, we show that with $\gamma \in \Gamma$ arbitrary and

$$X_{\gamma^2} = \{x \in R^n;\ R_{\gamma^2} x = x\} ,$$

one has $\operatorname{sgn} \det L_{\gamma^2} = \operatorname{sgn} \det L$, where $L_{\gamma^2} \equiv L|_{X_{\gamma^2}} \in GL(X_{\gamma^2})$ (as X_{γ^2} is a fixed point space, it is indeed invariant under L). To see this, note that X_{γ^2} is invariant under R_γ since γ and γ^2 commute. As R is orthogonal, the space $X^{\perp}_{\gamma^2}$ is invariant under R_γ, too. Therefore, relative to the splitting $R^n = X_{\gamma^2} \oplus X^{\perp}_{\gamma^2}$, R_γ has the block diagonal decomposition

$$R_\gamma = \begin{bmatrix} R_{\gamma 1} & 0 \\ 0 & R_{\gamma 2} \end{bmatrix} , \tag{3.7}$$

where $R_{\gamma 1} \in GL(X_{\gamma^2})$, $R_{\gamma 2} \in GL(X^{\perp}_{\gamma^2})$ are orthogonal. A key point is that $\pm 1 \notin \operatorname{Sp} R_{\gamma 2}$. Indeed, if there is $x \in X^{\perp}_{\gamma^2}$, $x \neq 0$, such that $R_{\gamma 2}x = \pm x$, then $R_\gamma x = \pm x$ and $R_{\gamma^2} x = x$. Thus, $x \in X_{\gamma^2}$, a contradiction.

Now, since $L(X_{\gamma^2}) = X_{\gamma^2}$, the corresponding block decomposition of L is

$$L = \begin{bmatrix} L_{11} & L_{12} \\ 0 & L_{22} \end{bmatrix} \tag{3.8}$$

where $L_{11} = L_{\gamma^2} \in GL(X_{\gamma^2})$, $L_{12} \in \mathcal{L}(X^{\perp}_{\gamma^2}, X_{\gamma^2})$ and $L_{22} \in GL(X^{\perp}_{\gamma^2})$. Writing $R_\gamma L = LR_\gamma$ (R-covariance of L) through (3.7) and (3.8) reveals that $R_{\gamma 2}L_{22} = L_{22}R_{\gamma 2}$. If $X^{\perp}_{\gamma^2} \neq \{0\}$, it then follows from Lemma 3.2 that $\det L_{22} \geq 0$ (i.e. $\det L_{22} > 0$) and hence $\operatorname{sgn} \det L = \operatorname{sgn} \det L_{11}$. The same relation is true

if $X^{\perp}_{\gamma^2} = \{0\}$, for then $L_{11} = L$. Recalling that $L_{11} = L_{\gamma^2}$ one finds sgn $\det L$ = sgn $\det L_{\gamma^2}$, as desired. Observe in passing that the relation $\det L_{22} > 0$ above does imply that $X_{\gamma^2} \neq \{0\}$ if $\det L < 0$.

Now, coming back to the more specific case when $\gamma \in N_\Gamma(\Gamma')$, the relation sgn $\det L_{\tilde{\Gamma}'}$ = sgn $\det L$ will follow from sgn $\det L_{\Gamma'}$ = sgn $\det L$ (since Γ' is an i.i.s.) and sgn $\det L_{\Gamma'}$ = sgn $\det L_{\tilde{\Gamma}'}$. To prove the latter equality, it suffices to use the result of the first step with the help of a few simple remarks. First, it is clear that this result remains valid if R^n is replaced by any n-dimensional inner product space X and R by a representation of Γ in GL(X) which is orthogonal for this inner product. Next, it is readily seen that

$$X_{\tilde{\Gamma}'} = X_{\Gamma'} \cap X_{\gamma^2} = \{x \in X_{\Gamma'};\ R_{\gamma^2}x = x\} . \tag{3.9}$$

The condition $\gamma \in N_\Gamma(\Gamma')$ ensures that $X_{\Gamma'}$ is invariant under R_γ , and $R'_\gamma \equiv R_{\gamma}|_{X_{\Gamma'}} \in GL(X_{\Gamma'})$ is orthogonal (orthogonality of a linear operator is not affected by restriction to an invariant subspace equipped with the induced inner product). As a result, $X_{\Gamma'}$ is invariant under R_δ for every $\delta \in \langle\gamma,\Gamma'\rangle$, the group generated by γ and Γ'. This yields that the mapping $R' : \delta \in \langle\gamma,\Gamma'\rangle \to R'_\delta \equiv R_\delta|_{X_{\Gamma'}}$ is an orthogonal representation of $\langle\gamma,\Gamma'\rangle$ in $GL(X_{\Gamma'})$ (with $R'_\delta = I$, $\forall\delta \in \Gamma'$). Moreover, it is trivial that $L_{\Gamma'} \equiv L|_{X_{\Gamma'}}$ is R'-covariant.

With all this, one may use the first step with Γ, R^n, L and R_γ being replaced by $\langle\gamma,\Gamma'\rangle, X_{\Gamma'}$, $L_{\Gamma'}$ and R'_γ , respectively. In view of (3.9), this

shows that sgn $\det L_{\Gamma'}$ = sgn $\det L_{\Gamma'}|_{X_{\tilde{\Gamma}'}}$. But $L_{\Gamma'}|_{X_{\tilde{\Gamma}'}} = L_{\tilde{\Gamma}'}$ is obvious, and we are done. In particular, if $\det L < 0$, then $X_{\Gamma'} \neq \{0\}$ and $\det L_{\Gamma'} < 0$ since Γ' is an i.i.s. That $X_{\tilde{\Gamma}'} \neq \{0\}$ in this case thus also follows from the first step. □

Theorem 3.1: Let Γ be a compact Lie group and $\Gamma' \subset \Gamma$ a m.i.i.s. Then, letting $N_\Gamma(\Gamma')$ denote the normalizer of Γ' in Γ, one has

(i) $$(N_\Gamma(\Gamma')^2) \subset \Gamma' \subset N_\Gamma(\Gamma') ,$$

where $(N_\Gamma(\Gamma')^2)$ is the subgroup of Γ generated by elements of the form γ^2, $\gamma \in N_\Gamma(\Gamma')$.

(ii) $$N_\Gamma(\Gamma')/\Gamma' \approx Z_2\times\cdots\times Z_2 \text{ (k factors, } 0 \leq k < \infty) .$$

Proof: (i) From Lemma 3.3, $<\gamma^2,\Gamma'>$ is an i.i.s. of Γ for every $\gamma \in N_\Gamma(\Gamma')$. Obviously, $\Gamma' \subset <\gamma^2,\Gamma'>$ and maximality of Γ' shows that $\Gamma' = <\gamma^2,\Gamma'>$, whence $\gamma^2 \in \Gamma'$. This proves (i).

(ii) It is easily checked that the closure of an i.i.s. is an i.i.s., too. Thus, Γ' is closed by maximality and, in this case, $N_\Gamma(\Gamma')$ is closed. As a result, both Γ' and $N_\Gamma(\Gamma')$ are compact Lie groups, and so is the factor group $N_\Gamma(\Gamma')/\Gamma'$ as is well known. Let $\bar{\gamma} \in N_\Gamma(\Gamma')/\Gamma'$ and choose a representative $\gamma \in N_\Gamma(\Gamma')$ of $\bar{\gamma}$. From part (i), $\gamma^2 \in \Gamma'$ so that $\bar{\gamma}^2 = 1$. In summary, $N_\Gamma(\Gamma')/\Gamma'$ is a compact Lie group, all of whose elements $\bar{\gamma}$ satisfy $\bar{\gamma}^2 = 1$. Such a group must have dimension 0 (i.e. is finite) since it cannot contain a subgroup isomorphic to SO(2). On the other hand, a finite group G such that $g^2 = 1$, $\forall g \in G$ is abelian (write $(g_1g_2)^2 = 1$ to get $g_1g_2 = g_2^{-1}g_1^{-1} = g_2g_1$ since $g_1^2 = g_2^2 = 1$), hence the direct product of cyclic groups. A generator g of any such cyclic group is such that $g^2 = 1$, so that $(g) \approx Z_2$. This completes the

proof. □

Remark 3.1: Another interesting analogy between representation-dependent isotropy subgroups and intrinsic ones is evidenced by Theorem 3.1 if one observes that it remains valid if Γ' is replaced by any representation-dependent maximal isotropy subgroup with one-dimensional fixed point space. The proof of this assertion is elementary. Moreover, in this case, one has the stronger conclusion that $N_\Gamma(\Gamma') = \Gamma'$ or $N_\Gamma(\Gamma')/\Gamma' \approx \mathbf{Z}_2$, (see [7, p. 79]). □

Corollary 3.1 (existence of nontrivial intrinsic isotropy subgroups): Let Γ be a compact Lie group which is not the direct product of copies of $\mathbf{Z}_2$. Then, Γ contains a nontrivial (maximal) intrinsic isotropy subgroup.

Proof: From Lemma 3.1, Γ contains a m.i.i.s. Γ'. Suppose $\Gamma' = \{1\}$, so that $N_\Gamma(\Gamma') = \Gamma$. Applying Theorem 3.1 (ii), one finds that Γ is the direct product of copies of $\mathbf{Z}_2$, a contradiction. □

Corollary 3.1 is optimal, because $\Gamma = Z_2 \times \cdots \times Z_2$ (k factors) does not have any nontrivial i.i.s. This follows immediately from the more general:

Theorem 3.2: Let Γ be a compact Lie group, possibly finite, and let $\Gamma' \subset \Gamma$ be an intrinsic isotropy subgroup. Then, $\Gamma' \subset (\Gamma^2)$ (the group generated by elements of the form γ^2, $\gamma \in \Gamma$).

Proof: The group (Γ^2) is a normal subgroup of Γ, and it is also closed. For the latter point, observe first that the identity component Γ_0 of Γ is contained in (Γ^2): this is obvious if Γ is finite and, if $\dim \Gamma \geq 1$, every element $\gamma \in \Gamma_0$ is contained in a maximal torus T (see [1] or [2]) hence can be written in the form $\gamma = \delta^2$ with $\delta \in T$. Thus, $(\Gamma^2)/\Gamma_0$ is finite, hence closed, and $(\Gamma^2) = \pi^{-1}((\Gamma^2)/\Gamma_0)$ where $\pi : \Gamma \to \Gamma/\Gamma_0$ is the projection, is closed, too

(recall that the identity component of a Lie group is a closed normal subgroup).

From the above, (Γ^2) is a (compact) Lie group normal in Γ, so that the factor group $\Gamma/(\Gamma^2)$ is also a compact Lie group. Moreover, $\Gamma/(\Gamma^2) \approx (\Gamma/\Gamma_0)/((\Gamma^2)/\Gamma_0)$ is finite. Finally, if $\bar{\gamma} \equiv \pi(\gamma) \in \Gamma/(\Gamma^2)$, then $\bar{\gamma}^2 = \pi(\gamma^2) = 1$. It follows that $\Gamma/(\Gamma^2) \approx \mathbf{Z}_2\times\cdots\times\mathbf{Z}_2$ (k factors, $0 \leq k < \infty$). If $k = 0$, one has $(\Gamma^2) = \Gamma$ and the result is trivial. We henceforth assume $k \geq 1$.

Given $\gamma_0 \in \Gamma$, $\gamma_0 \notin (\Gamma^2)$, one has $\bar{\gamma}_0 \equiv \pi(\gamma_0) \neq 1$. We shall prove that γ_0 can lie in no i.i.s. of Γ by exhibiting a representation R of Γ in $GL(R) \approx R - \{0\}$ such that $R_{\gamma_0} = -1$: any representation in GL(R) admits covariant linear isomorphisms with negative determinant, but the fixed point space relative to R of any subgroup Γ' containing γ_0 reduces to $\{0\}$ and hence Γ' cannot be an i.i.s. of Γ.

From the isomorphism $\Gamma/(\Gamma^2) \approx \mathbf{Z}_2\times\cdots\times\mathbf{Z}_2$ and $\bar{\gamma}_0 \neq 1$, we shall obtain R in the form $R = \bar{R}\circ\pi$ and $\bar{R}$ is a representation of $\mathbf{Z}_2\times\cdots\times\mathbf{Z}_2$ in GL(R) such that $\bar{R}_{\bar{\gamma}_0} = -1$. Existence of $\bar{R}$ is elementary: set $\bar{\gamma}_0 = (\epsilon_{01},\ldots,\epsilon_{0k})$ with $\epsilon_{0i} = \pm 1$. As $\bar{\gamma}_0 \neq 1$, there is at least one index i_0 such that $\epsilon_{0i_0} = -1$. The mapping:

$$\bar{\gamma} = (\epsilon_1,\ldots,\epsilon_k) \in \mathbf{Z}_2\times\cdots\times\mathbf{Z}_2 \to \begin{cases} -1 & \text{if } \epsilon_{i_0} = -1 , \\ 1 & \text{if } \epsilon_{i_0} = 1 , \end{cases}$$

provides the desired representation $\bar{R}$. □

In the next three theorems, we establish properties of a general nature that will turn out to be essential in the next section.

Theorem 3.3: let Γ be a compact Lie group and $\Gamma' \subset \Gamma$ an i.i.s. of Γ. Next, let $\Gamma'' \subset \Gamma'$ be a subgroup of Γ' such that Γ'' is normal in Γ (e.g. if Γ'' is a characteristic subgroup of Γ' and Γ' is normal in Γ). Then, Γ'' is an i.i.s. of Γ.

Proof: Consider any representation R of Γ in $GL(R^n)$ and let $L \in GL(R^n)$ be a R-covariant linear isomorphism. Since Γ' is an i.i.s. of Γ, one has $\mathrm{sgn}\ \det L_{\Gamma'} = \mathrm{sgn}\ \det L$ ($L_{\Gamma'} \equiv L_{|X_{\Gamma'}}$). On the other hand, $X_{\Gamma'} \subset X_{\Gamma''}$ since $\Gamma'' \subset \Gamma'$, and $X_{\Gamma''}$ is stable under L since it is a fixed point space. That $L_{\Gamma''} \equiv L_{|X_{\Gamma''}} \in GL(X_{\Gamma''})$ is obvious.

As Γ'' is normal in Γ, $X_{\Gamma''}$ is also stable under R_γ, $\forall \gamma \in \Gamma$, and $R'' : \gamma \in \Gamma \to R''_\gamma \equiv R_{\gamma_{|X_{\Gamma''}}} \in GL(X_{\Gamma''})$ is clearly a representation of Γ in $GL(X_{\Gamma''})$. It is trivial to check that R-covariance of L implies R''-covariance of $L_{\Gamma''}$ and that the fixed point space of Γ' relative to R'' is nothing but $X_{\Gamma'}$ (recall $X_{\Gamma'} \subset X_{\Gamma''}$). Accordingly, using the fact that Γ' is an i.i.s. of Γ (1) one finds that $\mathrm{sgn}\ \det(L_{\Gamma''_{|X_{\Gamma'}}}) = \mathrm{sgn}\ \det L_{\Gamma''}$. But $L_{\Gamma''_{|X_{\Gamma'}}} = L_{\Gamma'}$. This yields $\mathrm{sgn}\ \det L_{\Gamma''} = \mathrm{sgn}\ \det L$ by transitivity, and we are done. □

A partial converse of Theorem 3.3 is as follows.

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(1) here, we use the fact that $X_{\Gamma''} \approx R^n$ for some n; this identification is harmless to our definition of an i.i.s.

Theorem 3.4: *Let Γ be a compact Lie group and $\Gamma' \subset \Gamma$ a closed normal i.i.s. of Γ (i.e. an i.i.s. of Γ which is a closed normal subgroup of Γ). Then, the canonical projection $\pi : \Gamma \to \Gamma/\Gamma'$ induces a bijection between the i.i.s's of Γ containing Γ' and the i.i.s's of Γ/Γ'. Moreover, this bijection preserves normality and maximality.*

Proof: It is well known ("correspondence theorem") that π induces a normality-preserving bijection between the subgroups of Γ containing Γ' and the subgroups of Γ/Γ'. Therefore, given a subgroup $\tilde{\Gamma}'$ of Γ such that $\Gamma' \subset \tilde{\Gamma}'$, it suffices to show that $\tilde{\Gamma}'$ is an i.i.s. of Γ if and only if $\pi(\tilde{\Gamma}')$ is an i.i.s. of Γ/Γ'. Once this is done, the fact that the bijection preserves maximality is obvious. Note that closedness of Γ' suffices to ensure that Γ/Γ' is a (compact) Lie group.

Suppose first that $\tilde{\Gamma}'$ *is an* i.i.s. of Γ. Given any representation $\bar{R}$ of Γ/Γ' in $GL(R^n)$, one obtains a representation R of Γ through $R = \bar{R}\circ\pi$. Obviously, every $\bar{R}$-covariant linear isomorphism $L \in GL(R^n)$ is R-covariant, too. Since $\tilde{\Gamma}'$ is an i.i.s. of Γ, one has $\operatorname{sgn} \det L_{\tilde{\Gamma}'} = \operatorname{sgn} \det L$, where $L_{\tilde{\Gamma}'} = L_{|X_{\tilde{\Gamma}'}}$ and $X_{\tilde{\Gamma}'} = \{x \in R^n;\ R_\gamma x = x,\ \forall\gamma \in \tilde{\Gamma}'\}$. As $R_\gamma = \bar{R}_{\bar{\gamma}}$ with $\bar{\gamma} = \pi(\gamma)$ by definition of R, it follows that $X_{\tilde{\Gamma}'} = \bar{X}_{\pi(\tilde{\Gamma}')} = \{x \in R^n;\ \bar{R}_{\bar{\gamma}} x = x,\ \forall\bar{\gamma} \in \pi(\tilde{\Gamma}')\}$. The relation $\operatorname{sgn} \det L_{\tilde{\Gamma}'} = \operatorname{sgn} \det L$ is then exactly the one needed to conclude that $\pi(\tilde{\Gamma}')$ is an i.i.s. of Γ/Γ'.

Suppose now that $\pi(\tilde{\Gamma}')$ is an i.i.s. of Γ/Γ' and let R be any representation of Γ in $GL(R^n)$. Since Γ' is an i.i.s. of Γ, one has $\operatorname{sgn} \det L_{\Gamma'} = \operatorname{sgn} \det L$ for every covariant linear isomorphism $L \in GL(R^n)$ where, as usual, $L_{\Gamma'} = L_{|X_{\Gamma'}}$. Because Γ' is normal in Γ, the fixed point space $X_{\Gamma'}$

is invariant under R_γ , $\forall\gamma \in \Gamma$. As a result, R can also be viewed as a representation of Γ in $GL(X_{\Gamma'})$. Since R_γ reduces to I on $X_{\Gamma'}$ whenever $\gamma \in \Gamma'$, it follows that R factors through a representation $\bar{R}$ of Γ/Γ' in $GL(X_{\Gamma'})$. Obviously, $L_{\Gamma'}$ is $\bar{R}$-covariant. Set

$$\bar{X}_{\pi(\tilde{\Gamma}')} = \{x \in X_{\Gamma'} \ ; \ \bar{R}_{\bar{\gamma}} x = x, \forall\bar{\gamma} \in \pi(\tilde{\Gamma}')\} \ .$$

As $\Gamma' \subset \tilde{\Gamma}'$ and $\bar{R}_{\bar{\gamma}} = R_\gamma$ whenever $\bar{\gamma} = \pi(\gamma)$, it follows that the above space $\bar{X}_{\pi(\tilde{\Gamma}')}$ is nothing but the fixed point space $X_{\tilde{\Gamma}'} = \{x \in R^n; R_\gamma x = x, \ \forall\gamma \in \tilde{\Gamma}'\}$. Using that $\pi(\tilde{\Gamma}')$ is an i.i.s. of Γ/Γ', this remark thus yields $\text{sgn}\ \det L_{\Gamma'} = \text{sgn}\ \det L_{\Gamma'}|_{X_{\tilde{\Gamma}'}}$. But $L_{\Gamma'}|_{X_{\tilde{\Gamma}'}} = L_{\tilde{\Gamma}'}$ is obvious and hence $\text{sgn}\ \det L_{\tilde{\Gamma}'} = \text{sgn}\ \det L$ by transitivity. □

The following result is trivial, but quite useful

<u>Theorem 3.5</u>: <u>Let Γ be a compact Lie group and $\Gamma' \subset \Gamma$ an i.i.s. of Γ. Then, Γ' remains an i.i.s of any compact Lie group $\tilde{\Gamma}$ containing Γ as a subgroup</u>.

<u>Proof</u>: Every representation of $\tilde{\Gamma}$ yields one of Γ by restriction. □

4. <u>Examples of intrinsic isotropy subgroups</u>.

From Theorem 3.5, a possible approach for finding intrinsic isotropy subgroups is to characterize maximal ones. Indeed, every m.i.i.s. of a group Γ is an i.i.s. (not maximal, in general) of any group larger than Γ. Finding maximal isotropy subgroups is made easier through the criterion given in Theorem 3.1, not valid for nonmaximal i.i.s's. Also, from Theorem 3.2, any m.i.i.s. must be a subgroup of the group (Γ^2). There follows the natural question as to when (Γ^2) is a m.i.i.s. of Γ. In particular, $(\Gamma^2) = \Gamma$ if Γ is

finite and $|\Gamma|$ is odd Our first theorem settles the problem of finding those finite groups of odd order which are i.i.s's of themselves, in the simplest possible way.

Theorem 4.1: Every finite group Γ of odd order is an i.i.s. of itself.

Proof: We proceed in three steps:

(i) The theorem is true if Γ is abelian. Indeed, let Γ' be a m.i.i.s. of Γ. Since Γ is abelian, $N_\Gamma(\Gamma') = \Gamma$ and hence Γ/Γ' is the direct product of $0 \leq k < \infty$ copies of Z_2 (Theorem 3.1). But $|\Gamma/\Gamma'|$ is odd, whence $k = 0$ and $\Gamma' = \Gamma$.

(ii) The theorem is true if there is a normal i.i.s. Γ' of Γ such that Γ/Γ' is abelian. Indeed, $|\Gamma/\Gamma'|$ is odd so that Γ/Γ' is an i.i.s. of itself by step (i), and it suffices to apply *Theorem* 3.4.

(iii) The theorem is true in general. The celebrated theorem of Feit and Thompson states that every group of odd order is solvable. Let then

$$\Gamma \supset D\Gamma \supset \cdots \supset D^k\Gamma = \{1\} ,$$

be the derived series. All the terms of the series and their factors have odd order and the factors are abelian. In particular, $D^{k-1}\Gamma$ is an i.i.s. of itself from step (i), hence an i.i.s. of $D^{k-2}\Gamma$ (Theorem 3.5). Moreover, $D^{k-1}\Gamma$ is normal in $D^{k-2}\Gamma$ and $D^{k-2}\Gamma/D^{k-1}\Gamma$ is abelian. Thus, $D^{k-2}\Gamma$ is an i.i.s. of itself from step (ii). Clearly, the same procedure can be repeated up until one finds that Γ is an i.i.s. of itself. □

Theorem 4.1 is consistent with (but much stronger than) the known property that a group of odd order affords only one irreducible representation of real or quaternionic type, i.e. the trivial one (see e.g. [8]). Also, a

direct proof (not using the Feit-Thompson theorem, but also not completely algebraic) of Theorem 4.1 can be given but is more technical.

Remark 4.1: From Theorem 3.3, Γ is certainly not an i.i.s. of itself when $(\Gamma^2) \neq \Gamma$. When Γ is finite, this requires $|\Gamma|$ to be even but, conversely, $|\Gamma|$ even does not imply $(\Gamma^2) \neq \Gamma$. For instance, all finite simple groups (all of even order from the Feit-Thompson theorem) satisfy $(\Gamma^2) = \Gamma$. The same relation holds with $\Gamma = \mathbf{A}_4$ (solvable) of order 12. This leaves open the possibility that a finite group Γ of even order by an i.i.s. of itself. However, one can show that this never happens: no finite group of even order is an i.i.s. of itself. This result relies more upon group representation theory and will not be proved here. □

For our next theorem, we recall that a Dedekind group is a group all of whose subgroups are normal. This includes abelian groups. For a complete characterization of the Dedekind groups, see e.g. [18].

Theorem 4.2: Let Γ be a Dedekind compact Lie group: Then, the subgroup (Γ^2) is an i.i.s. of Γ (hence, (Γ^2) is the unique m.i.i.s. of Γ).

Proof: Consider a m.i.i.s. Γ' of Γ. As Γ is a Dedekind group, Theorem 3.1 yields $(\Gamma^2) \subset \Gamma' \subset \Gamma$. But $\Gamma' \subset (\Gamma^2)$ from Theorem 3.3 whence $\Gamma' = (\Gamma^2)$. □

From Theorem 4.2, one finds that $\Gamma' = \mathbf{Z}_2$ is an i.i.s. of $\mathbf{Z}_4$ or Q_8 (the quaternion group). Let us emphasize that this is not in contradiction with $\mathbf{Z}_2$ not being an i.i.s. of itself since the notion of i.i.s. strongly depends upon the ambient group. More generally, $\Gamma' = \mathbf{Z}_n$ is an i.i.s. of $\mathbf{Z}_{2n}$ and also of D_{4n} since $Z_{2n} \subset \mathbf{Z}_{4n}$ (but dihedral groups are not Dedekind). If n is odd, $\mathbf{Z}_n$ is an i.i.s. of itself (from Theorems 4.1 or 4.2), hence of D_{2n}.

Looking more closely, one may notice that $Z_n = (D_{4n}^2)$, and $Z_n = (D_{2n}^2)$ if n is odd. Hence, despite D_{2n} is not Dedekind, (D_{2n}^2) is an i.i.s. of D_{2n} be n even or odd. It is also a m.i.i.s. since (Γ^2) is maximal whenever it is an i.i.s. of Γ from Theorem 3.3.

Remark 4.2: The above observation that (D_{2n}^2) is always an i.i.s. of D_{2n} despite D_{2n} is not Dedekind strongly suggests that Theorem 4.2 is not optimal. Indeed, using again heavier material from group theory, one can show that (Γ^2) is an i.i.s. in every finite group Γ having the property that the elements of odd order in Γ form a subgroup Γ_{odd}. This class includes all supersolvable groups, hence the dihedral ones. Moreover, the situation described above is optimal in the sense that the following three statements are equivalent (with $|\Gamma| < \infty$)

(i) The elements of odd order in Γ form a subgroup,

(ii) For every subgroup G of Γ with $|G|$ even, one has $(G^2) \neq G$,

(iii) For every subgroup G of Γ, (G^2) is an i.i.s. of G.

That (i) $\Rightarrow$ (iii) follows from the above and the remark that the property that the elements of odd order form a subgroup is inherited by all subgroups. Also, (iii) $\Rightarrow$ (ii) is easy upon using the result in Remark 4.1 and (ii) $\Rightarrow$ (i) is elementary Finally, a statement generalizing Theorem 4.2 for arbitrary compact Lie groups can be derived without difficulty by applying the above results to the finite group Γ/Γ_0 (Γ_0 = identity component of Γ; recall $\Gamma_0 \subset (\Gamma^2)$ was noticed in the proof of Theorem 3.3). □

The problem of characterizing maximal intrinsic isotropy subgroups of finite groups becomes quite challenging for groups not in the class described

in Remark 4.2. The simplest example, the alternating group $\mathbf{A}_4$, shows that the situation is drastically different; viewing $\mathbf{A}_4$ as a subgroup of SO(3) and using a result from the next section (and keeping Theorem 3.5 in mind) it is a simple matter to check that all four subgroups of $\mathbf{A}_4$ isomorphic to $\mathbf{Z}_3$ are m.i.i.s's of $\mathbf{A}_4$. It is more difficult to show, essentially via Theorem 7.1 later, that its three subgroups isomorphic to $\mathbf{Z}_2$ are <u>also</u> m.i.i.s's (and no others exist). This shows that m.i.s.s's need not be conjugate and need not even have the same number of elements.

The following corollary to Theorem 4.2 is especially instructive:

<u>Corollary 4.1</u>: <u>Let</u> Γ <u>be a compact Lie group with</u> $\dim \Gamma \geq 1$. <u>Then, every (maximal) torus of</u> Γ <u>is an i.i.s. of</u> Γ.

<u>Proof</u>: From Theorem 3.5 it suffices to prove that a torus is an i.i.s. of itself. But if Γ is a torus, one has $(\Gamma^2) = \Gamma$ and the conclusion follows from Theorem 4.2 since Γ is abelian. □

As we shall see in the next section, the maximal torus SO(2) in SO(3) is also a m.i.i.s. of SO(3). It is even the only one up to conjugation. This makes it natural to ask whether maximal tori of a connected compact Lie group Γ are always m.i.i.s's of Γ (connectedness makes the question quite different from the one for finite groups examined in Remark 4.2). Interestingly, it follows from Theorem 3.1 that the answer is in the <u>negative</u>. Indeed, let $T \subset \Gamma$ be a maximal torus. Then the factor group $N_\Gamma(T)/T$ is known as the Weyl group W of Γ. If T is a m.i.i.s. of Γ, then $W = \mathbf{Z}_2^k$ for some $0 \leq k < \infty$ according to Theorem 3.1. But, for instance, one has $W = \mathbf{G}_n$ - the group of permutations τ of $\{-n,\dots,-1,1,\dots,n\}$ such that $\tau(-j) = -\tau(j)$, $j = -1,\dots,\pm n$ - when $\Gamma = SO(2n+1)$ (see e.g. [2]) and $\mathbf{G}_n \neq \mathbf{Z}_2^k$ if $n > 1$. Other counterexamples are

obtained with $\Gamma = SO(2n)$ for $n > 2$ (because $W = SG_n$, the group of even permutations of G_n) or with $\Gamma = U(n)$ or $SU(n)$ (because $W = S_n$, the symmetric group of order n). Thus, it appears that it is the exception rather than the rule that a maximal torus be also a m.i.i.s. of a compact connected Lie group. In fact, a result much stronger (apparently at least, see Remark 4.3) than Corollary 4.1 is true, namely

Theorem 4.3: Let Γ be a compact Lie group. Then every compact, connected solvable subgroup Γ' of Γ is an i.i.s. of Γ.

Proof: From Theorem 3.5, it suffices to show that a compact, connected solvable Lie group is an i.i.s. of itself. To do this, one may use the same three steps as in the proof of Theorem 4.1, the case when Γ is abelian (i.e. a torus) being solved e.g. through Corollary 4.1. In the second step, suppose also that Γ' is closed: connectedness of Γ/Γ' follows from continuity of the projection $\pi : \Gamma \to \Gamma/\Gamma'$ since Γ is connected. For the third step, observe that the terms of the derived series may be replaced by their closures without affecting normality of the embeddings or commutativity of the factors (see e.g. [19, p. 203]). □

Remark 4.3: Despite theorem 4.3 is more general than Corollary 4.1, we know of no explicit example of a compact connected solvable Lie group which is not a torus. □

5. Application.

In this short section, we shall briefly discuss a few consequences of the combination of Theorem 2.1 with the results of the previous section. As in

Section 2, we shall denote by $g(=g(\mu,x)) : R\times R^n \to R^n$ a $\mathscr{C}^1$ mapping covariant under some representation R of a compact Lie group Γ (possibly finite) and satisfying the conditions $g(\mu,0) \equiv 0$ and $\det D_x g(\mu,0)$ changes sign as μ crosses 0.

Suppose first that $\Gamma \neq \{1\}$ is finite and is not the direct product of copies of Z_2. Then, it follows from Theorem 2.1 and Corollary 3.1 that some symmetries, independent of the representation R, must be preserved in some continuum of bifurcated solutions. Actually, the fact that a continuum of Γ'-symmetric solutions bifurcates for every (maximal) intrinsic isotropy subgroup Γ' of Γ is much stronger than this statement alone. But the theory does not ascertain that different continua correspond to different maximal intrinsic isotropy subgroups. This depends in part upon the representation R. In any case, if $|\Gamma|$ is odd, then solutions having full Γ-symmetry will bifurcate (see Theorem 4.1). If Γ is Dedekind or even more general, solutions having (Γ^2)-symmetry will bifurcate (see Theorem 4.2 and Remark 4.2). In any case, Γ can be replaced by any subgroup having either property.

A comment equally valid in the case when Γ is a general compact Lie group is that each m.i.i.s. is contained in a maximal isotropy subgroup relative to R when the representation R is fixed point free and admits covariant linear isomorphisms with negative determinant, e.g. in odd dimension since $-I$ works (as we shall see later, the converse is not true).

Some interesting conclusions can also be drawn in the case when Γ is a compact Lie group with $\dim \Gamma \geq 1$: combining Theorem 2.1 and Corollary 4.1 one finds at once that a continuum of nontrivial solutions bifurcates that preserves the symmetry of any given maximal torus of Γ. In particular, if Γ is connected, every element $\gamma \in \Gamma$ is contained in a maximal torus, so that

every symmetry of Γ is preserved in some continuum of bifurcated solutions. As an example, take $\Gamma = SO(2k)$ or $SO(2k+1)$, in which maximal tori have dimension k. More specifically, if $k = 1$ and $\Gamma = SO(2)$, one finds that a continuum of nontrivial solutions bifurcates that preserves the symmetry of the entire group (recall the standing hypothesis that $\det D_x g(\mu,0)$ changes sign, which is fundamental for the validity of this statement).

If $\Gamma = SO(3)$, every maximal torus is isomorphic to $SO(2)$. Thus, one finds that a continuum of axisymmetric solutions bifurcates. This is consistent with the conclusions obtained from the branching lemmas of Vanderbauwhede [21] or Cicogna [5] through calculation of the maximal isotropy subgroups and the dimension of their fixed point space in the particular case when R is the irreducible representation of $SO(3)$ in the space V_ℓ of spherical harmonics of degree ℓ (see [7]). As $\dim V_\ell = 2\ell+1$ and $D_x G(\mu,0) = \alpha(\mu)I$ if $g(\mu,\cdot)$ is R-covariant from absolute irreducibility of R, $\det D_x g(\mu,0)$ changes sign if and only if $\alpha(\mu)$ does the same. This is precisely the case when bifurcation of axisymmetric solutions can be derived from the branching lemmas, because the dimension of the fixed point space of $SO(2)$ relative to R turns out to be one. It is noteworthy that the considerably more general result obtained by combining Theorem 2.1 and Corollary 4.1 does not require any calculation of representation-dependent isotropy subgroups, let alone that of the dimension of their fixed point space.

When $\Gamma = SO(3)$, one can show that the unique m.i.i.s. of Γ (up to conjugation) is $SO(2)$. Indeed, let $\Gamma' \subset \Gamma$ be a m.i.i.s. If R denotes the unique irreducible representation of $SO(3)$ in $GL(V_1)$, then R is fixed point free (because of irreducibility and $\dim V_1 > 1$). In fact, $\dim V_1 = 3$ is odd and hence Γ' must be contained in a maximal isotropy subgroup of $SO(3)$ relative to

R. But SO(2) is the unique such maximal isotropy subgroup up to conjugation (see [7]), whence $\Gamma' \subset SO(2)$. As SO(2) is a torus, hence an i.i.s., maximality of Γ' yields that $\Gamma' = SO(2)$.

In turn, this may be used to show that not every maximal isotropy subgroup of a fixed point free representation contains a m.i.i.s. For this, it suffices to take R as the unique irreducible representation of SO(3) in $GL(V_3)$, which has the alternating group $\mathbf{A}_4$ as a maximal isotropy subgroup although, obviously, $SO(2) \not\subset \mathbf{A}_4$. Via Theorem 3.4, this relation also shows that $\mathbf{A}_4$ cannot be an i.i.s. of itself, and the same is true with $\mathbf{A}_5$ or $\mathbf{S}_4$ for the same reason. More generally, the i.i.s's of $\mathbf{D}_{2n}$, $\mathbf{A}_4$, $\mathbf{A}_5$ and $\mathbf{S}_4$ must be subgroup of some $SO(2) \subset SO(3)$. This remark easily yields [illegible] statements made in Section 4 regarding the m.i.i.s's of $\mathbf{D}_{2n}$ and $\mathbf{A}_4$.

<u>Remark 5.1</u>: The following observation may have some practical value: let $\Gamma' \subset \Gamma$ be an i.i.s. and set $g_{\Gamma'}(\mu,\cdot) = g(\mu,\cdot)_{|X_{\Gamma'}} : X_{\Gamma'} \to X_{\Gamma'}$. Then, by definition of an i.i.s., $\det D_x g(\mu,\cdot)$ changes sign as μ crosses 0 if and only if $\det D_x g_{\Gamma'}(\mu,\cdot)$ does the same. The latter may be easier to check since $X_{\Gamma'} \subset R^n$, and may also be easier to check with some Γ' than others, although the sign change will occur or not with all intrinsic isotropy subgroups simultaneously. □

6. A complementing theorem.

It was observed earlier that finite groups isomorphic to a direct product of copies of Z_2 have the peculiarity of containing no nontrivial i.i.s. In view of Theorem 2.1, this may appear as a rather negative property of such groups. On the other hand, it is of course hopeless to look for a

representation-independent theorem guaranteeing existence of nontrivial solutions preserving a (nontrivial) symmetry of a group in a class that includes the group $\Gamma = Z_2$. This is because $-1 \in Z_2$ can always be realized as $-I$ through a representation of Z_2 in $GL(R^n)$. Our next result shows that, in some sense, Z_2 is the only compact Lie group to be excluded for the validity of such theorems, for every group $\Gamma \neq \{1\}$ of the form $\Gamma = Z_2 \times \cdots \times Z_2$ (k factors) either reduces to $Z_2 (k=1)$ or contains a subgroup isomorphic to $Z_2 \times Z_2$ (actually, it contains exactly $(2^k-1)(2^{k-1}-1)/3$ such subgroups). We retain all previous hypotheses made about the mapping g, most importantly its covariance under a representation R of a group Γ, and change of sign of $\det D_x g(\mu,0)$ as μ crosses 0.

Theorem 6.1: *Suppose that the group Γ contains a subgroup Γ' isomorphic to $Z_2 \times Z_2$, i.e. $\Gamma' = \{1,\gamma,\delta,\gamma\delta\}$ where $\gamma^2 = \delta^2 = 1$, $\gamma \neq 1$, $\delta \neq 1$, $\gamma \neq \delta$ and $\gamma\delta = \delta\gamma$. Then, for at least one of $\sigma = \gamma,\delta$ or $\gamma\delta$, there are σ-symmetric nontrivial solutions to $g=0$ bifurcating from $(0,0)$ and these solutions contain a continuum (here, σ-symmetric means (σ)-symmetric, where $(\sigma) = \{1,\sigma\}$ is the subgroup generated by σ).*

Proof: One may assume that $\Gamma' = \Gamma$ with no loss of generality, and hence that the representation R is orthogonal. In this proof, we let σ denote either of γ,δ or $\gamma\delta$ and, as usual, we set

$$X_\sigma = \{x \in R^n ; R_\sigma x = x\} .$$

The space R^n splits as the direct sum

$$R^n = X_\sigma \oplus Y_\sigma , \tag{6.1}$$

where $Y_\sigma = X_\sigma^\perp$.

Since $\sigma^2 = 1$, the eigenvalues of R_σ are ± 1 and hence the operator R_σ

reduces to I (resp. $-I$) on X_σ (resp. Y_σ). Because the group $\Gamma \approx Z_2 \times Z_2$ is abelian, both spaces X_σ and Y_σ are invariant not only under R_σ, but also under $R_{\sigma'}$, for every other $\sigma' = \gamma, \delta$ or $\gamma\delta$. This remark may be used as follows: take for instance $\sigma = \delta$ and write $x \in X_\gamma$ in the form

$$x = x_\delta + y_\delta \tag{6.2}$$

according to (6.1) with $\sigma = \delta$. Since $R_\gamma x = x$, one gets $x = R_\gamma x_\delta + R_\gamma y_\delta$. Since X_δ and Y_δ are invariant under R_γ, it follows from uniqueness of the decomposition (6.2) that $R_\gamma x_\delta = x_\delta$, $R_\gamma y_\delta = y_\delta$. In other words, $x_\delta \in X_\gamma \cap X_\delta$ and $y_\delta \in X_\gamma \cap Y_\delta$. This also reads

$$X_\gamma = (X_\gamma \cap X_\delta) \oplus (X_\gamma \cap Y_\delta) \ . \tag{6.3}$$

Similarly

$$Y_\gamma = (Y_\gamma \cap X_\delta) \oplus (Y_\gamma \cap Y_\delta) \ . \tag{6.4}$$

These relations are essential and they do rely on the properties of invariance mentioned above because, in general, a relation such as $Z \cap (X \oplus Y) = (Z \cap X) \oplus (Z \cap Y)$ is (obviously) not true.

Combining (6.1) with (6.3) and (6.4), one finds

$$R^n = (X_\gamma \cap X_\delta) \oplus (X_\gamma \cap Y_\delta) \oplus (Y_\gamma \cap X_\delta) \oplus (Y_\gamma \cap Y_\delta) \ . \tag{6.5}$$

Each of the four spaces in (6.4) is invariant under R_γ and R_δ, and hence under $R_{\gamma\delta} = R_\gamma R_\delta$ as well. More precisely, using the notation I(X) for the identity of the space X it follows from the definition of X_σ and Y_σ that, relative to (6.5), the operators R_γ and R_δ have the block diagonal decompositions

$$R_\gamma = \begin{pmatrix} I(X_\gamma \cap X_\delta) & & & \\ & I(X_\gamma \cap Y_\delta) & & \\ & & -I(Y_\gamma \cap X_\delta) & \\ & & & -I(Y_\gamma \cap Y_\delta) \end{pmatrix}$$

and

$$R_\delta = \begin{bmatrix} I(X_\gamma \cap X_\delta) & & & \\ & -I(X_\gamma \cap Y_\delta) & & \\ & & I(Y_\gamma \cap X_\delta) & \\ & & & -I(Y_\gamma \cap Y_\delta) \end{bmatrix} .$$

Hence, $R_{\gamma\delta}$ has the block diagonal decomposition

$$R_{\gamma\delta} = \begin{bmatrix} I(X_\gamma \cap X_\delta) & & & \\ & -I(X_\gamma \cap Y_\delta) & & \\ & & -I(Y_\gamma \cap X_\delta) & \\ & & & I(Y_\gamma \cap Y_\delta) \end{bmatrix} .$$

These expressions for R_γ, R_δ and $R_{\gamma\delta}$ make it clear that relation (6.3) is a particular case of

$$\begin{cases} X_\gamma = (X_\gamma \cap X_\delta) \oplus (X_\gamma \cap Y_\delta) , \\ X_\delta = (X_\gamma \cap X_\delta) \oplus (Y_\gamma \cap X_\delta) , \\ X_{\gamma\delta} = (X_\gamma \cap X_\delta) \oplus (Y_\gamma \cap Y_\delta) . \end{cases} \tag{6.6}$$

Let now $L \in \mathcal{L}(R^n)$ be such that $LR_\sigma = R_\sigma L$ for $\sigma = \gamma, \delta, \gamma\delta$. Then, the spaces X_σ and Y_σ are invariant under L (the latter because R_σ reduces to $-I$ on Y_σ and linearity of L) and hence the same thing is true of all possible intersections of spaces X_σ and Y_σ. In particular, relative to (6.5), L has the block diagonal decomposition

$$L = \begin{bmatrix} L_1 & & & \\ & L_2 & & \\ & & L_3 & \\ & & & L_4 \end{bmatrix} . \tag{6.7}$$

Combining (6.6) and (6.7), one finds

$$L|_{X_\gamma} = \begin{bmatrix} L_1 & \\ & L_2 \end{bmatrix}, \quad L|_{X_\delta} = \begin{bmatrix} L_1 & \\ & L_3 \end{bmatrix}, \quad L|_{X_{\gamma\delta}} = \begin{bmatrix} L_1 & \\ & L_4 \end{bmatrix}. \tag{6.8}$$

Now, let $L = L(\mu) = D_x g(\mu,0)$. In an obvious notation, it follows from (6.7) that one of the determinants $\det L_i(\mu)$, $1 \leq i \leq 4$, changes sign as μ crosses 0. Suppose first that $\det L_1(\mu)$ changes sign. Then, $\det L_i(\mu)$ does not change sign for some $2 \leq i \leq 4$ (because if all four $\det L_i(\mu)$ change sign, then $\det L(\mu)$ does not, a contradiction). Examination of (6.8) immediately reveals that $\det L(\mu)|_{X_\sigma}$ changes sign for $\sigma = \gamma$ or δ or $\gamma\delta$. Suppose now that $\det L_1(\mu)$ does not change sign, so that one among $\det L_i(\mu)$, $2 \leq i \leq 4$, does. Again, from (6.8), one of $\det L(\mu)|_{X_\sigma}$ changes sign for $\sigma = \gamma$ or δ or $\gamma\delta$. Existence of γ-, δ- or $\gamma\delta$-symmetric nontrivial solutions to $g=0$ bifurcating from (0,0) follows with the same topological degree argument as in the proof of Theorem 2.1. □

The basic difference between Theorems 2.1 and 6.1 is that, in the latter, the preserved symmetry (or symmetries) may depend upon the representation. Theorem 6.1 may be used separately, or jointly with Theorem 2.1 to show that more symmetries than expected are preserved in some bifurcated solutions. To illustrate the latter point, suppose that Γ is a compact Lie group and let $\Gamma' \subset \Gamma$ be a m.i.i.s. From Theorem 3.1, one knows that $N_\Gamma(\Gamma')/\Gamma' \approx Z_2 \times \cdots \times Z_2$ (k factors, $0 \leq k < \infty$). If $k \geq 2$, Theorem 6.1 implies that a continuum of nontrivial solutions to $g=0$ bifurcates that preserves <u>more</u> than Γ'-symmetry. Indeed, the fixed point space $X_{\Gamma'}$ of Γ' relative to R is invariant under the representation of $N_\Gamma(\Gamma')$ deduced from R, and this representation factors through Γ' as a representation of $N_\Gamma(\Gamma')/\Gamma' \approx Z_2 \times \cdots \times Z_2$ (since R_γ reduces to I on $X_{\Gamma'}$, $\forall \gamma \in \Gamma'$). Now, $g(\mu,\cdot)|_{X_{\Gamma'}}$ is obviously covariant under this

representation, and $\det D_x g(\mu,0)|_{X_{\Gamma'}}$ changes sign as μ crosses 0 if $\det D_x g(\mu,0)$ does so since Γ' is an i.i.s. This makes Theorem 6.1 available with $g(\mu,\cdot)|_{X_{\Gamma'}}$ replacing $g(\mu,\cdot)$ and yields the desired result at once.

Remark 6.1: The exceptional role of the group Z_2 was already noticed in [16] where one finds the following result: let Γ be a finite group and let $\Gamma' \subset \Gamma$ be any subgroup with $|\Gamma'| \geq 3$. Assuming that the representation R is faithful, $\det D_x g(\mu,0)$ changes sign as μ crosses 0 and $x = 0$ is an isolated solution to $g(0,x) = 0$, there is $\gamma \in \Gamma'$, $\gamma \neq 1$, such that γ-symmetric nontrivial solutions to $g=0$ bifurcate from $(0,0)$. In [16], the proof uses a generalized version of Borsuk's theorem. Here, one can get the same conclusion as a result of Theorems 2.1 and 6.1 without assuming that R is faithful nor that $x=0$ is an isolated solution to $g(0,x) = 0$. Indeed, let $|\Gamma'| = p_1^{\alpha_1} \cdots p_m^{\alpha_m}$ where the p_i's are distinct primes. The standard theorem of Cauchy (see e.g. [12]) ascertains that there is $\gamma_i \in \Gamma'$ with order p_i. If $p_i \neq 2$ for some i, say $i = 1$, the group (γ_1) has odd order, hence is an i.i.s. of Γ' (Theorem 4.1) and the conclusion follows from Theorem 2.1. It remains to consider the case $|\Gamma'| = 2^\alpha$ with $\alpha \geq 2$ from $|\Gamma'| \geq 3$ by hypothesis. If so, Γ' contains a subgroup of order 4 by Sylow's generalization of Cauchy's theorem (see again [12]). This subgroup is isomorphic to either Z_4 or $Z_2 \times Z_2$. In the first case, one may use the fact that Z_2 is an i.i.s. of Z_4 (see Section 4) and Theorem 2.1. In the second case, Theorem 6.1 applies. The special case of Theorem 2.1 when Γ' is a maximal torus in a Lie group Γ with $\dim \Gamma \geq 1$ is also in [16] with the same restrictions on R and g. □

Remark 6.2: As Theorem 2.1, Theorem 6.1 remains valid if $g(\mu,\cdot)$ is (R,S)-covariant, where R and S are equivalent representations of Γ in $GL(R^n)$. □

7. The variational case.

We shall now assume that the mapping $g(\mu,\cdot)$ is a gradient, say $g(\mu,x) = \nabla J(\mu,x)$ where "∇" refers to the x-variable alone and $J : R\times R^n \to R$ is a $\mathcal{C}^2$ functional.

Recently, several proofs have appeared that a change of Morse index for $D_x g(\mu,0)$ as μ crosses 0 is a sufficient condition for bifurcation of non-trivial solutions to $g=0$ near $(0,0)$ (see e.g. [4], [11], or [15]; the three proofs have in common the use of Conley index and generalize a standard result of Krasnoselskii). The Morse index of a symmetric operator $L \in \mathcal{L}(R^n)$ is defined as the difference between the numbers of positive and negative eigenvalues (or, alternatively, the number of positive eigenvalues; the change of one is equivalent to the change of the other).

Clearly, a change of sign of $\det D_x g(\mu,0)$ implies a change of Morse index for $D_x g(\mu,0)$, but the converse is not true. In other words, for gradient operators, the condition that the Morse index changes is more general than a change of sign of the determinant. This is especially important when covariance under some group action is involved. For instance, let $n=2$, $\Gamma = SO(2)$ acting on R^2 through the standard representation $R_\gamma = \gamma \in GL(R^2)$, $\forall\gamma \in SO(2)$. This representation is absolutely irreducible, so that the only R-covariant linear isomorphisms are the multiples of I. Now, if $D_x g(\mu,0) = \alpha(\mu)I$, it is obvious that $\det D_x g(\mu,0) = \alpha^2(\mu)$ cannot change sign. In contrast, the Morse index changes if $\alpha(\mu)$ changes sign.

When covariance of $g(\mu,\cdot)$ under a certain representation is assumed, Theorem 2.1 is not valid under the more general condition that the Morse index of $D_x g(\mu,0)$ changes. Quite naturally, this remark leads to the question

whether any connection exists at all between preservation of symmetries and change of Morse index in bifurcation problems involving gradient operators. At this point in time, we have not obtained a general positive answer to this question. However, as we shall see below, some connection does exist since Theorem 6.1 remains true.

<u>Theorem 7.1</u>: <u>Let</u> R <u>be an orthogonal representation of</u> $\Gamma = Z_2 \times Z_2$ <u>in</u> $GL(R^n)$ <u>and suppose</u> $g(\mu,\cdot) = \nabla J(\mu,\cdot)$ <u>where the</u> $\mathcal{C}^2$ <u>functional</u> $J : R \times R^n \to R$ <u>is</u> R-<u>invariant</u> (<u>i.e.</u> $J(\mu,R_\sigma x) = J(\mu,x)$, $\forall(\mu,x) \in R \times R^n$, $\forall\sigma \in \Gamma$). <u>Then</u>, $g(\mu,\cdot)$ <u>is</u> R-<u>covariant and</u>, <u>if the Morse index of</u> $D_x g(\mu,0)$ <u>changes as</u> μ <u>crosses</u> 0, <u>there is</u> $\sigma \in \Gamma$, $\sigma \neq 1$, <u>such that</u> γ-<u>symmetric nontrivial solutions to</u> $g=0$ <u>bifurcate from</u> $(0,0)$.

<u>Note</u>: If R is not orthogonal, then g is $(R,(R^T)^{-1})$-covariant. In fact, it is not restrictive to assume that R is orthogonal since Γ is finite and hence orthogonality is true with a possibly different inner product on R^n: although this modifies "∇", the equation $\nabla J(\mu,\cdot) = 0$ still characterizes the critical points of $J(\mu,\cdot)$.

<u>Proof</u>: We shall use the notation of the proof of Theorem 6.1, in particular $L(\mu) = D_x g(\mu,0)$. From the decomposition (6.7) for $L(\mu)$, one finds that a change of Morse index for $L(\mu)$ implies a change of Morse index for at least one among $L_1(\mu),\cdots,L_4(\mu)$.

Because $L_1(\mu) = L(\mu)|_{X_\gamma \cap X_\delta} = \nabla[J(\mu,\cdot)|_{X_\gamma \cap X_\delta}]$, change of Morse index for $L_1(\mu)$ implies existence of bifurcated solutions in the space $R \times (X_\gamma \cap X_\delta)$, that is, bifurcated solutions preserving both γ-and δ-symmetry (thus $\gamma\delta$, too).

Suppose then that the Morse index of $L_1(\mu)$ does not change, hence that of $L_i(\mu)$ does for some $2 \leq i \leq 4$. From (6.8), the Morse index of $L(\mu)|_{X_\gamma}$ or

$L(\mu)|_{X_\delta}$ or $L(\mu)|_{X_{\gamma\delta}}$ changes. But

$$L(\mu)|_{X_\sigma} = \nabla[J(\mu,\cdot)|_{X_\sigma}] , \quad \sigma = \gamma,\delta,\gamma\delta ,$$

and hence either γ or δ or $\gamma\delta$-symmetric solutions bifurcate. □

Remark 7.1: Theorem 7.1 should not be confused with earlier results by Michel [14], later generalized by Smoller and Wasserman [20] and proving existence of bifurcated solutions preserving the symmetry of representation-dependent maximal isotropy subgroups of general compact Lie groups for gradient systems: as Theorems 2.1 and 6.1, Theorem 7.1 is independent of the representation. □

8. The Banach space setting.

The results of Sections 2, 6 and 7 may also be used for bifurcation problems in Banach spaces X and Y via Lyapunov-Schmidt reduction. The action of the group Γ is then accounted for by equivalent representations R and S in GL(X) and GL(Y), respectively. It is well known that Lyapunov-Schmidt reduction may be performed in a way that does not affect covariance, in that it is inherited by the reduced mapping. This makes Theorems 2.1 and 6.1 available at once. If then g denotes the reduced mapping, let us recall that change of sign of $\det D_x g(\mu,0)$ is independent of the reduction (see e.g. [9], [15] and [6]), thus an intrinsic property of the original problem. It should also be noted that the kernel of the representations R and S may be different from the kernel of the finite dimensional representations deduced from R and S that are relevant in the reduced problem. Typically, R and S will be faithful, but the deduced representations need not be so (this happens e.g. in "rotating waves" problems involving SO(2) or O(2)-covariance; see the example in [7] and also

[16]). In this case, one may first replace the group Γ by the factor group Γ/K where K is the (common) kernel of the deduced representations, a normal subgroup of Γ. Every bifurcated solution will then have K-symmetry, so that Theorem 2.1 or 6.1, when applicable, will provide bifurcated solutions exhibiting more than K-symmetry. However, if K is difficult to identify, one may always use Theorems 2.1 or 6.1 directly since their validity does not require faithfulness of the representations.

To use Theorem 7.1 when $X = Y$ is a Hilbert space, there are two additional difficulties. The first one is to make the hypothesis intrinsic that the Morse index changes. This is done in [4] and [11] using an eigenvalue crossing condition, and in [15] using an algebraic criterion. The second difficulty is that the variational structure of a problem is not necessarily inherited by the reduced mapping. In [15], this difficulty is overcome as follows: suppose that the problem is to solve $G(\mu,x) = 0$ with $G(\mu,\cdot) = \nabla J(\mu,\cdot)$ and $G(\mu,0) = 0$. This problem is equivalent to solving $\hat{G}(\mu,x) = 0$ where $\hat{G}(\mu,x) = M^*(\mu)G(\mu,M(\mu)x)$ and $M(\mu) \in GL(X)$ is arbitrary. Clearly, $\hat{G}(\mu,\cdot) = \nabla\hat{J}(\mu,\cdot)$ where $\hat{J}(\mu,x) = J(\mu,M(\mu)x)$.

When $D_xG(\mu,0) \in \mathcal{L}(X)$ is analytic in μ, one may choose $M(\mu) = U(\mu)$, an analytic family of orthogonal operators with the property that $N = \mathrm{Ker}\, D_x\hat{G}(0,0) = \mathrm{Ker}\, D_xG(0,0)$ is invariant under $D_x\hat{G}(\mu,0)$ for $|\mu|$ small enough. This suffices to ascertain that the reduced problem for $\hat{G} = 0$, relative to $X = N \oplus N^\perp$, is variational (see [15] for details).

Suppose now that $J(\mu,\cdot)$ is R-invariant (hence $G(\mu,\cdot)$ is R-covariant) where R is an orthogonal representation of some group Γ. Again, orthogonality of R is never restrictive if Γ is a compact Lie group. It turns out that $U(\mu)$ above is R-covariant. This is due to the fact that (following Kato [10]) $U(\mu)$

is characterized as the solution to

$$\frac{d}{d\mu} U(\mu) = Q(\mu)U(\mu) \ , \ U(0) = I \ ,$$

and $Q(\mu)$ is the commutator $[\frac{dP}{d\mu}(\mu), P(\mu)]$ where $P(\mu)$ is the orthogonal projection of X onto $(D_x G(\mu,0)(\text{Range } D_x G(0,0)))^{\perp}$ (this space is R-invariant from R-covariance of $D_x G(\mu,0)$, so that $P(\mu)$ and $Q(\mu)$ are R-covariant).

It follows that $\hat{J}(\mu,\cdot)$ is R-invariant, whence $\hat{G}(\mu,\cdot)$ is R-covariant. As a result, the reduced problem for $\hat{G} = 0$ relative to $X = N \oplus N^{\perp}$ is not only variational but also R-covariant. This suffices to make Theorem 7.1 available. In fact, following other arguments in [15], analyticity of $D_x G(\mu,0)$ can be weakened.

<u>Remark 6.1</u>: A further reduction is used in [15], where $U(\mu)$ above is replaced by $M(\mu) = S(\mu)U(\mu)$ and $S(\mu)$ is a suitable family of "diagonal" operators. But this is done only in the aim of exhibiting an algebraic criterion equivalent to the change of Morse index in the reduced problem. The choice $M(\mu) = U(\mu)$ suffices for the purpose of making the reduced problem variational, and there is no difficulty in checking that the aforementioned criterion is the same for $D_x G(\mu,0)$ and $D_x \hat{G}(\mu,0)$. □

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